

Optimal insider control of systems with delay

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11 October 2016

MSC(2010): 60H05; 60H07; 60H40; 60G57; 91B70; 93E20.

Keywords: Stochastic delay equation; optimal control; inside information; Donsker delta functional; stochastic maximum principles; time-advanced BSDE; optimal insider portfolio in a financial market with delay.

Abstract

We study the problem of optimal inside control of a stochastic delay equation driven by a Brownian motion and a Poisson random measure. We prove a sufficient and a necessary maximum principle for the optimal control when the trader from the beginning has inside information about the future value of some random variable related to the system.

The results are applied to the problem of finding the optimal insider portfolio in a financial market where the risky asset price is given by a stochastic delay equation.

1 Introduction

In this paper we consider an insider's optimal control problem for a stochastic process $X(t) = X(t, Z) = X^u(t, Z)$ defined as the solution of a stochastic differential delay equation of the form

$$\begin{cases} dX(t) = dX(t, Z) = b(t, X(t, Z), Y(t, Z), u(t, Z))dt + \sigma(t, X(t, Z), Y(t, Z), u(t, Z))dB(t) \\ \quad + \int_{\mathbb{R}} \gamma(t, X(t, Z), Y(t, Z), u(t, Z), \zeta) \tilde{N}(dt, d\zeta), \quad 0 \leq t \leq T \\ X(t) = \xi(t), \quad -\delta \leq t \leq 0 \end{cases}$$

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³This research was carried out with support of the Norwegian Research Council, within the research project Challenges in Stochastic Control, Information and Applications (STOCONINF), project number 250768/F20.

where

$$Y(t, Z) = X(t - \delta, Z), \quad (1.1)$$

$\delta > 0$ being a fixed constant (the delay).

Here $B(t)$ and $\tilde{N}(dt, d\zeta)$ is a Brownian motion and an independent compensated Poisson random measure, respectively, jointly defined on a filtered probability space $(\Omega, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$ satisfying the usual conditions. $T > 0$ is a given constant. We refer to [ØS1] for more information about stochastic calculus for Itô-Lévy processes.

The process $u(t, Z) = u(t, x, z)_{z=Z}$ is our insider control process, where Z is a given \mathcal{F}_{T_0} -measurable random variable for some $T_0 > 0$, representing the inside information available to the controller.

We assume that the inside information is of *initial enlargement* type. Specifically, we assume that the inside filtration \mathbb{H} has the form

$$\mathbb{H} = \{\mathcal{H}_t\}_{0 \leq t \leq T}, \text{ where } \mathcal{H}_t = \mathcal{F}_t \vee \sigma(Z) \quad (1.2) \quad \{\text{eq1.1}\}$$

for all t , where Z is a given \mathcal{F}_{T_0} -measurable random variable, for some $T_0 > 0$ (constant). Here and in the following we use the right-continuous version of \mathbb{H} , i.e. we put $\mathcal{H}_t = \mathcal{H}_{t+} = \bigcap_{s>t} \mathcal{H}_s$.

We also assume that the *Donsker delta functional* of Z exists (see below). This assumption implies that the Jacod condition holds, and hence that $B(\cdot)$ and $N(\cdot, \cdot)$ are semimartingales with respect to \mathbb{H} . See e.g. [DØ2] for details. We assume that the value at time t of our insider control process $u(t)$ is allowed to depend on both Z and \mathcal{F}_t . In other words, $u(\cdot)$ is assumed to be \mathbb{H} -adapted, such that $u(\cdot, z)$ is \mathbb{F} -adapted for each $z \in \mathbb{R}$.

Let \mathbb{U} denote the set of admissible control values. We assume that the functions

$$\begin{aligned} b(t, x, y, u, z) &= b(t, x, y, u, z, \omega) : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \Omega \mapsto \mathbb{R} \\ \sigma(t, x, y, u, z) &= \sigma(t, x, y, u, z, \omega) : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \Omega \mapsto \mathbb{R} \\ \gamma(t, x, y, u, z, \zeta) &= \gamma(t, x, y, u, z, \zeta, \omega) : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \mathbb{R} \times \Omega \mapsto \mathbb{R} \end{aligned} \quad (1.3)$$

are given bounded C^1 functions with respect to x, y and u and adapted processes in (t, ω) for each given x, y, u, z, ζ . Let \mathcal{A} be a given family of admissible \mathbb{H} -adapted controls u . The *performance functional* $J(u)$ of a control process $u \in \mathcal{A}$ is defined by

$$J(u) = \mathbb{E} \left[\int_0^T f(t, X(t, Z), u(t, Z), Z) dt + g(X(T, Z), Y(T, Z), Z) \right], \quad (1.4) \quad \{\text{eq1.4}\}$$

where

$$\begin{aligned} f(t, x, u, z) &: [0, T] \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \mapsto \mathbb{R} \\ g(x, z) &: \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R} \end{aligned} \quad (1.5)$$

are given bounded functions, C^1 with respect to x and u . The functions f and g are called the *profit rate* and *terminal payoff*, respectively. For completeness of the presentation we allow these functions to depend explicitly on the future value Z also, although this would not be the typical case in applications. But it could be that f and g are influenced by the future value Z directly through the action of an insider, in addition to being influenced indirectly through the control process u and the corresponding state process X . The problem we consider is the following:

Problem 1.1 Find $u^* \in \mathcal{A}$ such that

$$\sup_{u \in \mathcal{A}} J(u) = J(u^*). \quad (1.6) \quad \{\text{eq1.5}\}$$

2 The Donsker delta functional

To study this problem we adapt the technique of the paper [DØ1] to the SDE with delay situation. For the convenience of the reader we first recall briefly the definition and basic properties of the Donsker delta functional:

Definition 2.1 Let $Z : \Omega \rightarrow \mathbb{R}$ be a random variable which also belongs to $(\mathcal{S})^*$. Then a continuous functional

$$\delta_Z(\cdot) : \mathbb{R} \rightarrow (\mathcal{S})^* \quad (2.1) \quad \{\text{donsker}\}$$

is called a Donsker delta functional of Z if it has the property that

$$\int_{\mathbb{R}} g(z) \delta_Z(z) dz = g(Z) \quad a.s. \quad (2.2) \quad \{\text{donsker pr}\}$$

for all (measurable) $g : \mathbb{R} \rightarrow \mathbb{R}$ such that the integral converges.

For example, consider the special case when Z is a first order chaos random variable of the form

$$Z = Z(T_0); \text{ where } Z(t) = \int_0^t \beta(s) dB(s) + \int_0^t \int_{\mathbb{R}} \psi(s, \zeta) \tilde{N}(ds, d\zeta), \text{ for } t \in [0, T_0] \quad (2.3) \quad \{\text{eq2.5}\}$$

for some deterministic functions $\beta \neq 0, \psi$ such that

$$\int_0^{T_0} \{\beta^2(t) + \int_{\mathbb{R}} \psi^2(t, \zeta) \nu(d\zeta)\} dt < \infty \text{ a.s.} \quad (2.4)$$

and for every $\epsilon > 0$ there exists $\rho > 0$ such that

$$\int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} e^{\rho \zeta} \nu(d\zeta) < \infty.$$

This condition implies that the polynomials are dense in $L^2(\mu)$, where $d\mu(\zeta) = \zeta^2 d\nu(\zeta)$. It also guarantees that the measure ν integrates all polynomials of degree ≥ 2 . In this case it is well known (see e.g. [MØP], [DiØ1], Theorem 3.5, and [DØP], [DiØ2]) that the Donsker delta functional exists in $(\mathcal{S})^*$ and is given by

$$\begin{aligned}\delta_Z(z) &= \frac{1}{2\pi} \int_{\mathbb{R}} \exp^\diamond \left[\int_0^{T_0} \int_{\mathbb{R}} (e^{ix\psi(s,\zeta)} - 1) \tilde{N}(ds, d\zeta) + \int_0^{T_0} ix\beta(s) dB(s) \right. \\ &\quad \left. + \int_0^{T_0} \left\{ \int_{\mathbb{R}} (e^{ix\psi(s,\zeta)} - 1 - ix\psi(s,\zeta)) \nu(d\zeta) - \frac{1}{2} x^2 \beta^2(s) \right\} ds - ixz \right] dx, \quad (2.5)\end{aligned}$$

where \exp^\diamond denotes the Wick exponential. Moreover, we have for $t < T_0$

$$\begin{aligned}\mathbb{E}[\delta_Z(z)|\mathcal{F}_t] &= \frac{1}{2\pi} \int_{\mathbb{R}} \exp \left[\int_0^t \int_{\mathbb{R}} ix\psi(s, \zeta) \tilde{N}(ds, d\zeta) + \int_0^t ix\beta(s) dB(s) \right. \\ &\quad \left. + \int_t^{T_0} \int_{\mathbb{R}} (e^{ix\psi(s,\zeta)} - 1 - ix\psi(s,\zeta)) \nu(d\zeta) ds - \int_t^{T_0} \frac{1}{2} x^2 \beta^2(s) ds - ixz \right] dx. \quad (2.6)\end{aligned}$$

If D_t and $D_{t,\zeta}$ denotes the *Hida-Malliavin derivative* at t and t, ζ with respect to B and \tilde{N} , respectively, we have

$$\begin{aligned}\mathbb{E}[D_t \delta_Z(z)|\mathcal{F}_t] &= \frac{1}{2\pi} \int_{\mathbb{R}} \exp \left[\int_0^t \int_{\mathbb{R}} ix\psi(s, \zeta) \tilde{N}(ds, d\zeta) + \int_0^t ix\beta(s) dB(s) \right. \\ &\quad \left. + \int_t^{T_0} \int_{\mathbb{R}} (e^{ix\psi(s,\zeta)} - 1 - ix\psi(s,\zeta)) \nu(d\zeta) ds - \int_t^{T_0} \frac{1}{2} x^2 \beta^2(s) ds - ixz \right] ix\beta(t) dx \quad (2.7)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[D_{t,z} \delta_Z(z)|\mathcal{F}_t] &= \frac{1}{2\pi} \int_{\mathbb{R}} \exp \left[\int_0^t \int_{\mathbb{R}} ix\psi(s, \zeta) \tilde{N}(ds, d\zeta) + \int_0^t ix\beta(s) dB(s) \right. \\ &\quad \left. + \int_t^{T_0} \int_{\mathbb{R}} (e^{ix\psi(s,\zeta)} - 1 - ix\psi(s,\zeta)) \nu(d\zeta) ds - \int_t^{T_0} \frac{1}{2} x^2 \beta^2(s) ds - ixz \right] (e^{ix\psi(t,z)} - 1) dx. \quad (2.8)\end{aligned}$$

For more information about the Donsker delta functional, Hida-Malliavin calculus and their properties, see [DØ1].

From now on we assume that Z is a given random variable which also belongs to $(\mathcal{S})^*$, with a Donsker delta functional $\delta_Z(z) \in (\mathcal{S})^*$ satisfying

$$\mathbb{E}[\delta_Z(z)|\mathcal{F}_T] \in \mathbf{L}^2(\mathcal{F}_T, P) \quad (2.9)$$

and

$$\mathbb{E} \left[\int_0^T (\mathbb{E}[D_t \delta_Z(z)|\mathcal{F}_t])^2 dt \right] < \infty, \text{ for all } z. \quad (2.10)$$

3 Transforming the insider control problem to a related parametrized non-insider problem

Since $X(t)$ is \mathbb{H} -adapted, we get by using the definition of the Donsker delta functional $\delta_Z(z)$ of Z that

$$X(t) = X(t, Z) = X(t, z)_{z=Z} = \int_{\mathbb{R}} X(t, z) \delta_Z(z) dz \quad (3.1) \quad \{\text{eq1.6}\}$$

for some z -parametrized process $X(t, z)$ which is \mathbb{F} -adapted for each z .

Then, again by the definition of the Donsker delta functional we can write, for $0 \leq t \leq T$

$$\begin{aligned} X(t) &= \xi(0, Z) + \int_0^t [b(s, X(s), Y(s), u(s, Z), Z)] ds + \int_0^t \sigma(s, X(s), Y(s), u(s, Z), Z) dB(s) \\ &+ \int_0^t \int_{\mathbb{R}} \gamma(s, X(s), Y(s), u(s, Z), Z, \zeta) \tilde{N}(ds, d\zeta) \\ &= \xi(0, z)_{z=Z} + \int_0^t [b(s, X(s, z), Y(s, z), u(s, z), z)]_{z=Z} ds \\ &+ \int_0^t \sigma(s, X(s, z), Y(s, z), u(s, z), z)_{z=Z} dB(s) \\ &+ \int_0^t \int_{\mathbb{R}} \gamma(s, X(s, z), Y(s, z), u(s, z), z, \zeta)_{z=Z} \tilde{N}(ds, d\zeta) \\ &= \int_{\mathbb{R}} \xi(x, z) \delta_Z(z) dz + \int_0^t \int_{\mathbb{R}} [b(s, X(s, z), Y(s, z), u(s, z), z)] \delta_Z(z) dz ds \\ &+ \int_0^t \int_{\mathbb{R}} \sigma(s, X(s, z), Y(s, z), u(s, z), z) \delta_Z(z) dz dB(s) \\ &+ \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \gamma(s, X(s, z), Y(s, z), u(s, z), z, \zeta) \delta_Z(z) dz \tilde{N}(ds, d\zeta) \\ &= \int_{\mathbb{R}} \{ \xi(0, z) + \int_0^t [b(s, X(s, z), Y(s, z), u(s, z), z)] ds + \int_0^t \sigma(s, X(s, z), Y(s, z), u(s, z), z) dB(s) \\ &+ \int_0^t \int_{\mathbb{R}} \gamma(s, X(s, z), Y(s, z), u(s, z), z, \zeta) \tilde{N}(ds, d\zeta) \} \delta_Z(z) dz. \end{aligned} \quad (3.2) \quad \{\text{eq1.7}\}$$

Comparing (3.1) and (3.2) we see that (3.1) holds if we for each z choose $X(t, z)$ as the solution of the classical (but parametrized) SPDE

$$\begin{cases} dX(t, z) = [b(t, X(t, z), Y(t, z), u(t, z), z)] dt + \sigma(t, X(t, z), Y(t, z), u(t, z), z) dB(t) \\ \quad + \int_{\mathbb{R}} \gamma(t, X(t, z), Y(t, z), u(t, z), z, \zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T] \\ X(t, z) = \xi(t); \quad t \in [-\delta, 0] \end{cases} \quad (3.3) \quad \{\text{eq3.3}\}$$

As before let \mathcal{A} be the given family of admissible \mathbb{H} -adapted controls u . Then in terms of $X(t, z)$ the performance functional $J(u)$ of a control process $u \in \mathcal{A}$ defined in (1.4) gets the form

$$\begin{aligned}
J(u) &= \mathbb{E} \left[\int_0^T f(t, X(t, Z), u(t, Z), Z) dt + g(X(T, Z), Z) \right] \\
&= \mathbb{E} \left[\int_0^T \left(\int_{\mathbb{R}} f(t, X(t, z), u(t, z), z) \mathbb{E}[\delta_Z(z) | \mathcal{F}_t] dz \right) dt + \int_{\mathbb{R}} g(X(T, z), z) \mathbb{E}[\delta_Z(z) | \mathcal{F}_T] dz \right] \\
&= \int_{\mathbb{R}} j(u)(z) dz,
\end{aligned} \tag{3.4} \quad \{\text{eq0.13}\}$$

where

$$\begin{aligned}
j(u)(z) &:= \mathbb{E} \left[\int_0^T f(t, X(t, z), u(t, z), z) \mathbb{E}[\delta_Z(z) | \mathcal{F}_t] dt \right. \\
&\quad \left. + g(X(T, z), z) \mathbb{E}[\delta_Z(z) | \mathcal{F}_T] \right].
\end{aligned} \tag{3.5} \quad \{\text{eq1.5}\}$$

Thus we see that to maximize $J(u)$ it suffices to maximize $j(u)(z)$ for each value of the parameter $z \in \mathbb{R}$. Therefore Problem 1.1 is transformed into the problem

Problem 3.1 *For each given $z \in \mathbb{R}$ find $u^* = u^*(t, z) \in \mathcal{A}$ such that*

$$\sup_{u \in \mathcal{A}} j(u)(z) = j(u^*)(z). \tag{3.6} \quad \{\text{problem2}\}$$

4 A sufficient-type maximum principle

In this section we will establish a sufficient maximum principle for Problem 3.1.

Problem 3.1 is a stochastic control problem with a standard (albeit parametrized) stochastic partial differential equation (3.3) for the state process $X(t, z)$, but with a non-standard performance functional given by (3.5). We can solve this problem by a modified maximum principle approach, as follows:

Define the *Hamiltonian* $H : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{U} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times \Omega \rightarrow \mathbb{R}$ by

$$\begin{aligned}
H(t, x, y, u, z, p, q, r) &= H(t, x, y, u, z, p, q, r, \omega) \\
&= \mathbb{E}[\delta_Z(z) | \mathcal{F}_t] f(t, x, u, z) + b(t, x, y, u, z) p \\
&\quad + \sigma(t, x, y, u, z) q + \int_{\mathbb{R}} \gamma(t, x, y, u, z, \zeta) r(\zeta) \nu(d\zeta).
\end{aligned} \tag{4.1} \quad \{\text{eq4.1}\}$$

\mathcal{R} denotes the set of all functions $r(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ such that the last integral above converges. The quantities $p, q, r(\cdot)$ are called the *adjoint variables*. The *adjoint processes* $p(t, z), q(t, z), r(t, z, \zeta)$ are defined as the solution of the z -parametrized advanced backward stochastic differential equation (ABSDE)

$$\begin{cases} dp(t, z) = \mathbb{E}[\mu(t, z)|\mathcal{F}_t]dt + q(t, z)dB(t) + \int_{\mathbb{R}} r(t, z, \zeta)\tilde{N}(dt, d\zeta) & t \in [0, T] \\ p(T, z) = \frac{\partial g}{\partial x}(X(T, z))\mathbb{E}[\delta_Z(z)|\mathcal{F}_T] \end{cases}$$

where

$$\begin{aligned} \mu(t, z) = & -\frac{\partial H}{\partial x}(t, X(t, z), Y(t, z), u(t, z), p(t, z), q(t, z)) \\ & -\frac{\partial H}{\partial y}(t + \delta, X(t + \delta, z), Y(t + \delta, z), u(t + \delta, z), p(t + \delta, z), q(t + \delta, z))\mathbf{1}_{[0, T-\delta]}(t) \end{aligned} \quad (4.2)$$

We can now state the first maximum principle for our problem (3.6):

Theorem 4.1 [*Sufficient-type maximum principle*]

Let $\hat{u} \in \mathcal{A}$, and denote the associated solution of (3.3) and (4.2) by $\hat{X}(t, z)$ and $(\hat{p}(t, z), \hat{q}(t, z), \hat{r}(t, z, \zeta))$, respectively. Assume that the following hold:

1. $x \rightarrow g(x, z)$ is concave for all z
2. $(x, y, u) \rightarrow H(t, x, y, u, z, \hat{p}(t, z), \hat{q}(t, z), \hat{r}(t, z, \zeta))$ is concave for all t, z, ζ
3. $\sup_{w \in \mathbb{U}} H(t, \hat{X}(t, z), \hat{Y}(t, z), w, \hat{p}(t, z), \hat{q}(t, z), \hat{r}(t, z, \zeta))$
 $= H(t, \hat{X}(t, z), \hat{Y}(t, z), \hat{u}(t, z), \hat{p}(t, z), \hat{q}(t, z), \hat{r}(t, z, \zeta))$ for all t, z, ζ .

Then $\hat{u}(\cdot, z)$ is an optimal insider control for Problem 3.1.

Proof. By considering an increasing sequence of stopping times τ_n converging to T , we may assume that all local integrals appearing in the computations below are martingales and hence have expectation 0. See [ØS2]. We omit the details.

Choose arbitrary $u(\cdot, z) \in \mathcal{A}$, and let the corresponding solution of (3.3) and (4.2) be $X(t, z)$, $p(t, x, z)$, $q(t, x, z)$, $r(t, x, z, \zeta)$. For simplicity of notation we write $f(t) = f(t, X(t, z), u(t, z))$, $\hat{f}(t) = f(t, \hat{X}(t, z), \hat{u}(t, z))$ and similarly with $b, \hat{b}, \sigma, \hat{\sigma}$ and so on.

Moreover put

$$\hat{H}(t) = H(t, \hat{X}(t, z), \hat{Y}(t, z), \hat{u}(t, z), \hat{p}(t, z), \hat{q}(t, z), \hat{r}(t, z, \cdot)) \quad (4.3)$$

and

$$H(t) = H(t, X(t, z), Y(t, z), u(t, z), p(t, z), q(t, z), r(t, z, \cdot)) \quad (4.4)$$

In the following we write $\tilde{f} = f - \hat{f}$, $\tilde{b} = b - \hat{b}$, $\tilde{X} = X - \hat{X}$.

Consider

$$j(u(\cdot, z)) - j(\hat{u}(\cdot, z)) = I_1 + I_2,$$

where

$$I_1 = \mathbb{E}\left[\int_0^T \{f(t) - \hat{f}(t)\}\mathbb{E}[\delta_Z(z)|\mathcal{F}_t]dt\right], \quad I_2 = \mathbb{E}[(g(X(T, z), z) - g(\hat{X}(T, z), z))\mathbb{E}[\delta_Z(z)|\mathcal{F}_T]]. \quad (4.5) \quad \{\text{eq4.7}\}$$

By the definition of H and the concavity of H , we have

$$\begin{aligned}
I_1 &= \mathbb{E} \left[\int_0^T \{ H(t, z) - \hat{H}(t, z) - \hat{p}(t, z) \tilde{b}(t, z) - \hat{q}(t, z) \tilde{\sigma}(t, z) \right. \\
&\quad \left. - \int_{\mathbb{R}} \hat{r}(t, z, \zeta) \tilde{\gamma}(t, z, \zeta) \nu(d\zeta) \} dt \right] \\
&\leq \mathbb{E} \left[\int_0^T \left(\frac{\partial \hat{H}}{\partial x}(t, z) \tilde{X}(t, z) + \frac{\partial \hat{H}}{\partial y}(t, z) \tilde{Y}(t, z) \right. \right. \\
&\quad \left. \left. + \frac{\partial \hat{H}}{\partial u}(t, z) \tilde{u}(t, z) - \hat{p}(t, z) \tilde{b}(t, z) - \hat{q}(t, z) \tilde{\sigma}(t, z) \right. \right. \\
&\quad \left. \left. - \int_{\mathbb{R}} \hat{r}(t, z, \zeta) \tilde{\gamma}(t, z, \zeta) \nu(d\zeta) \right) dt \right] \tag{4.6} \quad \{\text{eq4.8}\}
\end{aligned}$$

Since g is concave with respect to x we have

$$\begin{aligned}
&(g(X(T, z), z) - g(\hat{X}(T, z), z)) \mathbb{E}[\delta_Z(z) | \mathcal{F}_T] \\
&\leq \frac{\partial g}{\partial x}(\hat{X}(T, z), z) \mathbb{E}[\delta_Z(z) | \mathcal{F}_T] (X(T, z) - \hat{X}(T, z)), \tag{4.7}
\end{aligned}$$

and hence

$$\begin{aligned}
I_2 &\leq \mathbb{E} \left[\frac{\partial g}{\partial x}(\hat{X}(T, z)) \mathbb{E}[\delta_Z(z) | \mathcal{F}_T] \tilde{X}(T, z) \right] = \mathbb{E}[\hat{p}(T, z) \tilde{X}(T, z)] \tag{4.8} \quad \{\text{eq4.11}\} \\
&= \mathbb{E} \left[\int_0^T \hat{p}(t, z) d\tilde{X}(t, z) + \int_0^T \tilde{X}(t, z) d\hat{p}(t, z) + \int_0^T d[\hat{p}, \tilde{X}]_t \right] \\
&= \mathbb{E} \left[\int_0^T \{ \hat{p}(t, z) \tilde{b}(t, z) + \tilde{X}(t, z) \mathbb{E}[\mu(t, z) | \mathbb{F}_t] \right. \\
&\quad \left. + \tilde{\sigma}(t, z) \hat{q}(t, z) + \int_{\mathbb{R}} \tilde{\gamma}(t, z, \zeta) \hat{r}(t, z, \zeta) \nu(d\zeta) \} dt \right].
\end{aligned}$$

Combining (4.6) and (4.8) we obtain using the fact $X(t) = \hat{X}(t) = \xi(t)$ for all $t \in [-\delta, 0]$

$$\begin{aligned}
j(u(\cdot, z)) - j(\hat{u}(\cdot, z)) &\leq \mathbb{E} \left[\int_0^T \left(\frac{\partial \hat{H}}{\partial x}(t, z) \tilde{X}(t, z) + \frac{\partial \hat{H}}{\partial y}(t, z) \tilde{Y}(t, z) \right. \right. \\
&\quad \left. \left. + \frac{\partial \hat{H}}{\partial u}(t, z) \tilde{u}(t, z) + \mu(t, z) \tilde{X}(t, z) \right) dt \right] \\
&= \mathbb{E} \left[\int_{\delta}^{T+\delta} \left\{ \frac{\partial \hat{H}}{\partial x}(t - \delta, z) + \frac{\partial \hat{H}}{\partial y}(t, z) \mathbf{1}_{[0, T]}(t) + \hat{\mu}(t - \delta, z) \right\} \tilde{Y}(t, z) dt \right. \\
&\quad \left. + \int_0^T \frac{\partial \hat{H}}{\partial u}(t, z) \tilde{u}(t, z) dt \right] \tag{4.9} \quad \{\text{eq4.11a}\}
\end{aligned}$$

where

$$\hat{\mu}(t - \delta, z) = \frac{\partial \hat{H}}{\partial x}(t - \delta, z) + \frac{\partial \hat{H}}{\partial y}(t, z) \mathbf{1}_{[0, T]}(t) \tag{4.10}$$

then

$$\begin{aligned}
j(u(\cdot, z)) - j(\hat{u}(\cdot, z)) &\leq \mathbb{E} \left[\int_0^T \frac{\partial \hat{H}}{\partial u}(t, z) \tilde{u}(t, z) dt \right] \\
&\leq \mathbb{E} \left[\int_0^T \mathbb{E} \left[\frac{\partial \hat{H}}{\partial u}(t, z) \tilde{u}(t, z) | \mathcal{F}_t \right] dt \right] \\
&\leq \mathbb{E} \left[\int_0^T \mathbb{E} \left[\frac{\partial \hat{H}}{\partial u}(t, z) | \mathcal{F}_t \right] \tilde{u}(t, z) dt \right] \\
&\leq 0
\end{aligned} \tag{4.11}$$

The last inequality holds because of the maximum condition of H . Hence $j(u) \leq j(\hat{u})$. Since $u \in \mathcal{A}$ was arbitrary, this shows that \hat{u} is optimal. \square

5 A necessary-type maximum principle

In some cases the concavity conditions of Theorem 4.1 do not hold. In such situations a corresponding necessary-type maximum principle can be useful. For this, instead of the concavity conditions we need the following assumptions about the set of admissible control values:

- A_1 . For all $t_0 \in [0, T]$ and all bounded \mathcal{H}_{t_0} -measurable random variables $\alpha(z, \omega)$, the control $\theta(t, z, \omega) := \mathbf{1}_{[t_0, T]}(t) \alpha(z, \omega)$ belongs to \mathcal{A} .

- A_2 . For all $u; \beta_0 \in \mathcal{A}$ with $\beta_0(t, z) \leq K < \infty$ for all t, z define

$$\delta(t, z) = \frac{1}{2K} \text{dist}((u(t, z), \partial \mathbb{U}) \wedge 1 > 0 \tag{5.1} \quad \{\text{delta}\}$$

and put

$$\beta(t, z) = \delta(t, z) \beta_0(t, z). \tag{5.2} \quad \{\text{beta}(t, z)\}$$

Then the control

$$\tilde{u}(t, z) = u(t, z) + a\beta(t, z); \quad t \in [0, T]$$

belongs to \mathcal{A} for all $a \in (-1, 1)$.

- A_3 . For all β as in (5.2) the derivative process

$$\chi(t, z) := \frac{d}{da} X^{u+a\beta}(t, z)|_{a=0}$$

exists, and belongs to $\mathbf{L}^2(\lambda \times \mathbf{P})$ and

$$\begin{cases} d\chi(t, z) = [\frac{\partial b}{\partial x}(t, z)\chi(t, z) + \frac{\partial b}{\partial y}(t, z)\chi(t - \delta, z) + \frac{\partial b}{\partial u}(t, z)\beta(t, z)]dt \\ + [\frac{\partial \sigma}{\partial x}(t, z)\chi(t, z) + \frac{\partial \sigma}{\partial y}(t, z)\chi(t - \delta, z) + \frac{\partial \sigma}{\partial u}(t, z)\beta(t, z)]dB(t) \\ + \int_{\mathbb{R}} [\frac{\partial \gamma}{\partial x}(t, z, \zeta)\chi(t, z) + \frac{\partial \gamma}{\partial y}(t, z, \zeta)\chi(t - \delta, z) + \frac{\partial \gamma}{\partial u}(t, z, \zeta)\beta(t, z)]\tilde{N}(dt, d\zeta) \\ \chi(t, z) = 0 \quad \forall t \in [-\delta, 0]. \end{cases} \quad (5.3) \quad \{\mathbf{d} \ \chi\}$$

Theorem 5.1 *[Necessary maximum principle]*

Let $\hat{u} \in \mathcal{A}$. Then the following are equivalent:

1. $\frac{d}{da}J((\hat{u} + a\beta)(\cdot, z))|_{a=0} = 0$ for all bounded $\beta \in \mathcal{A}$ of the form (5.2).
2. $\mathbb{E}[\frac{\partial H}{\partial u}(t, z)|\mathcal{F}_t]_{u=\hat{u}} = 0$ for all $t \in [0, T]$.

Proof. For simplicity of notation we write u instead of \hat{u} in the following.

By considering an increasing sequence of stopping times τ_n converging to T , we may assume that all local integrals appearing in the computations below are martingales and have expectation 0. See [ØS2]. We omit the details.

We can write

$$\frac{d}{da}J((u + a\beta)(\cdot, z))|_{a=0} = I_1 + I_2$$

where

$$I_1 = \frac{d}{da}\mathbb{E}\left[\int_0^T f(t, X^{u+a\beta}(t, z), u(t, z) + a\beta(t, z), z)\mathbb{E}[\delta_Z(z)|\mathcal{F}_t]dt\right]|_{a=0}$$

and

$$I_2 = \frac{d}{da}\mathbb{E}[g(X^{u+a\beta}(T, z), z)\mathbb{E}[\delta_Z(z)|\mathcal{F}_T]]|_{a=0}.$$

By our assumptions on f and g and by (4.1) we have

$$\begin{aligned} I_1 &= \mathbb{E}\left[\int_0^T \left\{\frac{\partial f}{\partial x}(t, z)\chi(t, z) + \frac{\partial f}{\partial y}(t, z)\chi(t - \delta, z) + \frac{\partial f}{\partial u}(t, z)\beta(t, z)\right\}\mathbb{E}[\delta_Z(z)|\mathcal{F}_t]dt\right] \\ &= \mathbb{E}\left[\int_0^T \left\{\frac{\partial H}{\partial x}(t, z) - \frac{\partial b}{\partial x}(t, z)p(t, z) - \frac{\partial \sigma}{\partial x}(t, z)q(t, z) - \int_{\mathbb{R}} \frac{\partial \gamma}{\partial x}(t, z, \zeta)r(t, z, \zeta)\nu(d\zeta)\right\}\chi(t, z)dt\right. \\ &\quad + \int_0^T \left\{\frac{\partial H}{\partial y}(t, z) - \frac{\partial b}{\partial y}(t, z)p(t, z) - \frac{\partial \sigma}{\partial y}(t, z)q(t, z) - \int_{\mathbb{R}} \frac{\partial \gamma}{\partial y}(t, z, \zeta)r(t, z, \zeta)\nu(d\zeta)\right\}\chi(t - \delta, z)dt \\ &\quad \left. + \int_0^T \frac{\partial f}{\partial u}(t, z)\beta(t, z)\mathbb{E}[\delta_Z(z)|\mathcal{F}_t]dt\right] \end{aligned} \quad (5.4) \quad \{\mathbf{iii1}\}$$

$$I_2 = \mathbb{E}[g'(X(T, z), z)\chi(T, z)\mathbb{E}[\delta_Z(z)|\mathcal{F}_T]] = \mathbb{E}[p(T, z)\chi(T, z)] \quad (5.5) \quad \{\mathbf{iii2}\}$$

By the Itô formula

$$\begin{aligned}
I_2 &= \mathbb{E}[p(T, z)\chi(T, z)] = \mathbb{E}\left[\int_0^T p(t, z)d\chi(t, z) + \int_0^T \chi(t, z)dp(t, z) + \int_0^T d[\chi, p](t, z)\right] \quad (5.6) \\
&= \mathbb{E}\left[\int_0^T p(t, z)\left\{\frac{\partial b}{\partial x}(t, z)\chi(t, z) + \frac{\partial b}{\partial y}(t, z)\chi(t - \delta, z) + \frac{\partial b}{\partial u}(t, z)\beta(t, z)\right\}dt\right. \\
&\quad - \int_0^T \chi(t, z)\mathbb{E}[\mu(t, z)|\mathcal{F}_t]dt \\
&\quad + \int_0^T q(t, z)\left\{\frac{\partial \sigma}{\partial x}(t, z)\chi(t, z) + \frac{\partial \sigma}{\partial y}(t, z)\chi(t - \delta, z) + \frac{\partial \sigma}{\partial u}(t, z)\beta(t, z)\right\}dt \\
&\quad \left. + \int_0^T \int_{\mathbb{R}} \left\{\frac{\partial \gamma}{\partial x}(t, z, \zeta)\chi(t, z) + \frac{\partial \gamma}{\partial y}(t, z, \zeta)\chi(t, z) + \frac{\partial \gamma}{\partial u}(t, z, \zeta)\beta(t, z)\right\}r(t, z, \zeta)\nu(\zeta)dt\right]
\end{aligned}$$

Summing (5.4) and (5.6) we get

$$\begin{aligned}
\frac{d}{da}J((u + a\beta)(\cdot, y))|_{a=0} &= I_1 + I_2 = \mathbb{E}\left[\int_0^T \chi(t, z)\left\{\frac{\partial H}{\partial x}(t, z) + \mu(t, z)\right\}dt\right. \\
&\quad \left. + \int_0^T \chi(t - \delta, z)\frac{\partial H}{\partial y}(t, z) + \frac{\partial H}{\partial u}(t, z)\beta(t, z)dt\right] \\
&= \mathbb{E}\left[\int_0^T \chi(t, z)\left\{\frac{\partial H}{\partial x}(t, z) - \frac{\partial H}{\partial x}(t, z) - \frac{\partial H}{\partial y}(t + \delta, z)\mathbf{1}_{[0, T-\delta]}(t)\right\}dt\right. \\
&\quad \left. + \int_0^T \chi(t - \delta, z)\frac{\partial H}{\partial y}(t, z) + \frac{\partial H}{\partial u}(t, z)\beta(t, z)dt\right] \\
&= \mathbb{E}\left[\int_0^T \frac{\partial H}{\partial u}(t, z)\beta(t, z)dt\right] \quad (5.7)
\end{aligned}$$

we conclude that

$$\frac{d}{da}J((u + a\beta)(\cdot, z))|_{a=0} = 0$$

if and only if $\mathbb{E}[\int_0^T \frac{\partial H}{\partial u}(t, z)\beta(t, z)dt] = 0$ for all bounded $\beta \in \mathcal{A}$ of the form (5.2).

In particular, applying this to $\beta(t, z) = \alpha(z, \omega)\mathbf{1}_{[s, T]}(t)$ where $\alpha(z, \omega)$ is bounded and \mathcal{F}_{t_0} measurable, $s \geq t_0$ we obtain

$$\mathbb{E}\left[\int_s^T \frac{\partial H}{\partial u}(t, z)dt\alpha dt\right] = 0 \quad (5.8)$$

Differentiating with respect to s , we get

$$\mathbb{E}\left[\frac{\partial H}{\partial u}(s, z)\alpha\right] = 0.$$

Since this holds for all $s \geq t_0$ and for all α , we conclude that

$$\mathbb{E}\left[\frac{\partial H}{\partial u}(s, z)|\mathcal{F}_{t_0}\right] = 0.$$

□

6 Optimal insider portfolio in a financial market with delay

As an application of the results above, consider a financial market with the following two investment possibilities:

(i) A risk free asset, with unit price $S_0(t) = 1$ for all times $t \geq 0$.

(ii) A risky asset, in which the investments have a delayed effect, in the following sense: If we at time t invest in this asset the fraction $\pi(t, Z)$ of the current wealth $X(t, Z)$, then we assume that the dynamics of the wealth $X(t, Z) = X^\pi(t, Z)$ is described by a stochastic delay equation of the form

$$\begin{cases} dX(t, Z) = \pi(t, Z)[\alpha_0(t)X(t - \delta, Z)dt + \beta_0(t)X(t, Z)dB(t)], & 0 \leq t \leq T \\ X(t, Z) = \xi(t), & -\delta \leq t \leq 0 \end{cases} \quad (6.1) \quad \{\text{eq6.1}\}$$

Here $\alpha_0(t)$ and $\beta_0(t)$ are given bounded adapted processes and ξ is a given bounded deterministic function. The performance functional is defined by

$$J(\pi) = \mathbb{E}[\log X(T, Z)] = \mathbb{E}\left[\int_{\mathbb{R}} \log X(T, z)\mathbb{E}[\delta_Z(z)|\mathcal{F}_T]dz\right] = \int_{\mathbb{R}} j(u)dz, \quad (6.2)$$

where

$$j(u) = j(u, z) = \mathbb{E}[\log X(T, z)\mathbb{E}[\delta_Z(z)|\mathcal{F}_T]]. \quad (6.3) \quad \{\text{eq8.2}\}$$

Let $\mathcal{A}_{\mathbb{H}}$ be the set of \mathbb{H} -adapted controls $\pi(t) = \pi(t, Z)$ such that there is a unique solution $X(t) = X(t, Z)$ of (6.1) with $X(T, Z) > 0$ a.s. Note that equation (6.1) can be solved inductively step by step in each interval $[k\delta, (k+1)\delta]$ for $k = 0, 1, 2, \dots$. We study the following problem:

Problem 6.1 Find $\pi^* \in \mathcal{A}_{\mathbb{H}}$ (called an optimal control) such that:

$$\sup_{\pi \in \mathcal{A}_{\mathbb{H}}} J(\pi) = J(\pi^*) \quad (6.4)$$

This is a problem of the type investigated in the previous sections, in the special case with no jumps and with controls $\pi(t, z)$, and we can apply the results in Theorem 5.1 to study it.

For studies of financial markets modelled by stochastic delay equations we refer to [AHMP1], [AHMP2] and [KMT].

The Hamiltonian (4.1) gets the form, with $u = \pi$,

$$H(t, x, y, \pi, p, q) = \pi \alpha_0 y p + \pi \beta_0 x q \quad (6.5)$$

while the BSDE (4.2) for the adjoint processes becomes,

$$\begin{cases} dp(t, z) &= -\{\pi(t, z)\beta_0(t)q(t, z) + \mathbb{E}[\alpha_0(t + \delta)\pi(t + \delta)p(t + \delta, z)\mathbf{1}_{[0, T-\delta]}(t)|\mathcal{F}_t]\}dt \\ &+ q(t)dB(t) \\ p(T, z) &= \frac{1}{X(T, z)}\mathbb{E}[\delta_Z(z)|\mathcal{F}_T] \end{cases}$$

The equation

$$\frac{\partial}{\partial \pi} H(t, X(t, z), Y(t, z), \pi, p(t, z), q(t, z)) = 0 \quad (6.6)$$

is equivalent to

$$\alpha_0(t)X(t - \delta, z)p(t, z) + \beta_0(t)X(t, z)q(t, z) = 0, \quad (6.7)$$

or

$$q(t, z) = -\psi(t, z)p(t, z), \quad (6.8)$$

where

$$\psi(t, z) = \frac{\alpha_0(t)X(t - \delta, z)}{\beta_0(t)X(t, z)}. \quad (6.9)$$

Combining this with (6.1) we get

$$\begin{aligned} d(p(t, z)X(t, z)) &= p(t, z)\pi(t, z)[\alpha_0(t)X(t - \delta, z)dt + \beta_0(t)X(t, z)dB(t)] \\ &+ X(t, z)\left[-\{\pi(t, z)\beta_0(t)q(t, z) + \mathbb{E}[\alpha_0(t + \delta)\pi(t + \delta)p(t + \delta, z)\mathbf{1}_{[0, T-\delta]}(t)|\mathcal{F}_t]\}dt + q(t)dB(t)\right] \\ &+ \pi(t, z)\beta_0(t)X(t, z)q(t, z)dt \end{aligned} \quad (6.10)$$

Define

$$V(t, z) = p(t, z)X(t, z) \quad (6.11) \quad \{\text{eq5.11}\}$$

and

$$W(t, z) = V(t, z)[\pi(t, z)\beta_0(t) - \psi(t, z)] \quad (6.12) \quad \{\text{eq5.12}\}$$

then

$$\begin{cases} dV(t, z) &= \{\psi(t, z)W(t, z) + \psi^2(t, z) - \mathbb{E}[(\psi(t + \delta, z)W(t + \delta) + \psi^2(t + \delta, z))\mathbf{1}_{[0, T-\delta]}(t)|\mathcal{F}_t]\}dt \\ &+ W(t, z)dB(t), \quad 0 \leq t \leq T \\ V(T) &= \mathbb{E}[\delta_Z(z)|\mathcal{F}_T]. \end{cases}$$

For given $\psi(t, z)$ this is a time-advanced BSDE in the unknown processes $V(t, z) = V_\psi(t, z)$ and $W(t, z) = W_\psi(t, z)$. It can be solved by backward induction, as in [ØSZ]. Then, solving (6.12) for $\pi(t, z)$ and evaluating at $z=Z$, we get the following result:

Theorem 6.2 Suppose an optimal insider portfolio $\pi^*(t, Z)$ of Problem 5.1 exists. Then it is given in feedback form by

$$\pi^*(t, Z) = \frac{\alpha_0(t)}{\beta_0^2(t)} \cdot \frac{X(t - \delta, Z)}{X(t, Z)} + \frac{W_\psi(t, Z)}{\beta_0(t)V_\psi(t, Z)}, \quad (6.13) \quad \{\text{eq5.13}\}$$

where (V, W) is the solution of the BSDE below (6.12).

Remark 6.3 (The no-delay case) Note that the above theorem also applies to the special case when there is no delay, i.e. $\delta = 0$. In this case we see that $V(t, z) = \mathbb{E}[\delta_Z(z)|\mathcal{F}_t]$ and $W(t, z) = D_t V(t, z) = \mathbb{E}[D_t \delta_Z(z)|\mathcal{F}_t]$, and (6.13) reduces to

$$\pi^*(t, Z) = \frac{\alpha_0(t)}{\beta_0^2(t)} + \frac{\mathbb{E}[D_t \delta_Z(z)|\mathcal{F}_t]_{z=Z}}{\beta_0(t)\mathbb{E}[\delta_Z(z)|\mathcal{F}_t]_{z=Z}}. \quad (6.14)$$

This result has been proved in [ØR] and [DØ1] by different methods.

7 Optimal insider portfolio in a financial market with delay (with jumps)

In this section we add jumps to the model discussed in Section 6. Thus we consider the following controlled stochastic delay equation:

$$\begin{cases} dX(t, z) = \pi(t, z)[\alpha_0(t)X(t - \delta, z)dt + \beta_0(t)X(t, z)dB(t) + \int_{\mathbb{R}} X(t, z)\gamma_0(t, \zeta)\tilde{N}(dt, d\zeta)], & 0 \leq t \leq T \\ X(t, z) = \xi(t), & -\delta \leq t \leq 0 \end{cases} \quad (7.1) \quad \{\text{eq1}\}$$

with performance functional given by

$$J(\pi) = \mathbb{E}[\log X(T, Z)] = \mathbb{E}\left[\int_{\mathbb{R}} \log X(T, z)\mathbb{E}[\delta_Z(z)|\mathcal{F}_T]dz\right] = \int_{\mathbb{R}} j(u)du, \quad (7.2)$$

where

$$j(u) = j(u, z) = \mathbb{E}[\log X(T, z)\mathbb{E}[\delta_Z(z)|\mathcal{F}_T]], \quad (7.3) \quad \{\text{eq8.2}\}$$

Let $\mathcal{A}_{\mathbb{H}}$ be the set of \mathbb{H} -adapted controls $\pi(t)$. We study the following problem:

Problem 7.1 Find $\pi^* \in \mathcal{A}_{\mathbb{H}}$ such that:

$$\sup_{\pi \in \mathcal{A}_{\mathbb{H}}} J(\pi) = J(\pi^*) \quad (7.4)$$

In this case the Hamiltonian (4.1) gets the form, with $u = \pi$,

$$H(t, x, y, \pi, p, q) = \pi \alpha_0(t) y p + \pi \beta_0(t) x q + \pi x \int_{\mathbb{R}} \gamma_0(t, \zeta) r(\zeta) \nu(d\zeta) \quad (7.5)$$

while the BSDE (4.2) for the adjoint processes becomes,

$$\begin{cases} dp(t, z) &= -\{\pi(t, z)[\beta_0(t)q(t, z) + \int_{\mathbb{R}} \gamma_0(t, \zeta)r(t, z, \zeta)\nu(d\zeta)] \\ &\quad + \mathbb{E}[\alpha_0(t + \delta)\pi(t + \delta)p(t + \delta, z)\mathbf{1}_{[0, T-\delta]}(t)|\mathcal{F}_t]\}dt \\ &\quad + q(t)dB(t) \\ p(T, z) &= \frac{1}{X(T, z)}\mathbb{E}[\delta_Z(z)|\mathcal{F}_T] \end{cases}$$

The equation

$$\frac{\partial}{\partial \pi} H(t, X(t, z), Y(t, z), \pi, p(t, z), q(t, z), r(t, z, \cdot)) = 0 \quad (7.6)$$

is equivalent to

$$\alpha_0(t)X(t - \delta, z)p(t, z) + \beta_0(t)X(t, z)q(t, z) + X(t, z) \int_{\mathbb{R}} \gamma_0(t, \zeta)r(t, z, \zeta)\nu(d\zeta) = 0. \quad (7.7) \quad \{\text{eq7.7}\}$$

Define

$$u(t, z) = p(t, z)X(t, z). \quad (7.8) \quad \{\text{eq0.7}\}$$

Then by the Itô formula we get

$$\begin{aligned} du(t, z) &= p(t, z)\pi(t, z)\alpha_0(t)Y(t, z)dt - X(t, z)\pi(t, z)\beta_0(t, z)q(t, z)dt \\ &\quad - \int_{\mathbb{R}} X(t, z)\pi(t, z)\gamma_0(t, \zeta)\nu(d\zeta)dt \\ &\quad - X(t, z)\mathbb{E}[\alpha_0(t + \delta)\pi(t + \delta)p(t + \delta, z)\mathbf{1}_{[0, T-\delta]}(t)|\mathcal{F}_t]dt \\ &\quad + p(t, z)\pi(t, z)\beta_0(t)X(t, z)dB(t) + X(t, z)q(t, z)dB(t) \\ &\quad + q(t, z)X(t, z)\pi(t, z)\beta_0(t)dt \\ &\quad + \int_{\mathbb{R}} \{(X(t, z) + \pi(t, z)\gamma_0(t, \zeta)X(t, z))(p(t, z) + r(t, z, \zeta)) \\ &\quad - p(t, z)X(t, z) - p(t, z)\gamma_0(t, \zeta)\pi(t, z)X(t, z) - X(t, z)r(t, z, \zeta)\}\nu(d\zeta)dt \\ &\quad + \int_{\mathbb{R}} \{(X(t, z) + \pi(t, z)\gamma_0(t, \zeta)X(t, z))(p(t, z) + r(t, z, \zeta)) - p(t, z)X(t, z)\}\tilde{N}(dt, d\zeta) \\ &= \left(p(t, z)\pi(t, z)\alpha_0(t)X(t - \delta, z)dt - X(t, z)\mathbb{E}[\alpha_0(t + \delta)\pi(t + \delta)p(t + \delta, z)\mathbf{1}_{[0, T-\delta]}(t)|\mathcal{F}_t] \right)dt \\ &\quad + \left(u(t, z)\beta_0(t)\pi(t, z) + X(t, z)q(t, z) \right)dB(t) \\ &\quad + \int_{\mathbb{R}} \left(u(t, z)\pi(t, z)\gamma_0(t, \zeta) + X(t, z)r(t, z, \zeta)(1 + \pi(t, z)\gamma_0(t, \zeta)) \right)\tilde{N}(dt, d\zeta), \end{aligned} \quad (7.9) \quad \{\text{eq0.8}\}$$

Put

$$\phi(t) := \phi(t, z) := \alpha_0(t) \frac{X(t - \delta)}{X(t)}, \quad (7.10) \quad \{\text{eq0.9}\}$$

and define

$$v(t, z) := u(t, z)\beta_0(t)\pi(t, z) + X(t, z)q(t, z) \quad (7.11) \quad \{\text{eq7.10}\}$$

and

$$w(t, z, \zeta) := u(t, z)\pi(t, z)\gamma_0(t, \zeta) + X(t, z)r(t, z, \zeta)(1 + \pi(t, z)\gamma_0(t, \zeta)). \quad (7.12) \quad \{\text{eq7.11}\}$$

Then (7.9) can be written

$$\begin{aligned} du(t, z) = & \left(\phi(t)\pi(t, z)u(t, z) - \mathbb{E}[\phi(t + \delta)\pi(t + \delta, z)u(t + \delta, z)\mathbf{1}_{[0, T - \delta]}(t)|\mathcal{F}_t] \right) dt \\ & + v(t, z)dB(t) + \int_{\mathbb{R}} w(t, z, \zeta)\tilde{N}(dt, d\zeta), \end{aligned} \quad (7.13) \quad \{\text{eq7.12}\}$$

and the first order condition (7.7) gets the form

$$\phi(t)u(t, z) + \beta_0(t)X(t, z)q(t, z) + X(t, z) \int_{\mathbb{R}} \gamma_0(t, \zeta)r(t, z, \zeta)\nu(d\zeta) = 0. \quad (7.14) \quad \{\text{eq7.13}\}$$

With $\phi(t)$, $u(t, z)$, $v(t, z)$, $w(t, z, \zeta)$ (and the coefficients $\alpha_0(t)$, $\beta_0(t)$, $\gamma_0(t, \zeta)$) given, the equations (7.11), (7.12) and (7.14) constitutes a coupled system of 3 equations in the 3 unknowns $\pi(t, z)$, $X(t, z)q(t, z)$, $X(t, z)r(t, z, \zeta)$.

To investigate this system further, we proceed as follows:

From (7.11) we get:

$$X(t, z)q(t, z) = v(t, z) - u(t, z)\beta_0(t)\pi(t, z), \quad (7.15) \quad \{\text{eq7.16}\}$$

and from (7.12) we get

$$X(t, z)r(t, z, \zeta) = \frac{w(t, z, \zeta) - u(t, z)\pi(t, z)\gamma_0(t, \zeta)}{1 + \gamma_0(t, \zeta)\pi(t, z)} \quad (7.16) \quad \{\text{eq7.17}\}$$

Substituting (7.15) and (7.17) into (7.14) we obtain the following equation for the optimal portfolio $\pi(t, z) = \hat{\pi}(t, z) = \hat{\pi}_{(u, v, w)}(t, z)$:

$$\beta_0^2(t)u(t, z)\hat{\pi}(t, z) - \int_{\mathbb{R}} \gamma_0(t, \zeta) \frac{w(t, z, \zeta) - u(t, z)\hat{\pi}(t, z)\gamma_0(t, \zeta)}{1 + \gamma_0(t, \zeta)\hat{\pi}(t, z)} \nu(d\zeta) = \phi(t, z)u(t, z) + \beta_0(t)v(t, z). \quad (7.17) \quad \{\text{eq7.17}\}$$

Substituting this into (7.13) we can conclude as follows:

Theorem 7.2 *Suppose an optimal portfolio for Problem 7.1 exists and there exists a unique solution $\hat{\pi}(t, z)$, $(u(t, z), v(t, z), w(t, z, \zeta))$ of the coupled system consisting of (7.17) and the BSDE*

$$\begin{aligned} du(t, z) = & \left(\phi(t)\hat{\pi}_{(u, v, w)}(t, z)u(t, z) - \mathbb{E}[\phi(t + \delta)\hat{\pi}_{(u, v, w)}(t + \delta, z)u(t + \delta, z)\mathbf{1}_{[0, T - \delta]}(t)|\mathcal{F}_t] \right) dt \\ & + v(t, z)dB(t) + \int_{\mathbb{R}} w(t, z, \zeta)\tilde{N}(dt, d\zeta), \end{aligned} \quad (7.18) \quad \{\text{eq7.18}\}$$

$$u(T, z) = \mathbb{E}[\delta_Z(z)|\mathcal{F}_T] \quad (7.19) \quad \{\text{eq7.19}\}$$

Then the optimal insider portfolio $\hat{\pi}(t)$ is given by (7.17).

Remark 7.3 Equations (7.18)-(7.19) constitute a time-advanced BSDE which can (in principle) be solved backwards by proceeding as in [ØSZ].

References

- [AHMP1] M. Arriojas, Y. Hu, S.-E. Mohammed and G.Pap: A delayed Black and Scholes formula. *Stochastic Analysis and Applications* 25 (2007), 471-492.
- [AHMP2] M. Arriojas, Y. Hu, S.-E. Mohammed and G.Pap: A delayed Black and Scholes formula II. arxiv: 0604641v1, 28 April 2006
- [DØ1] O. Draouil and B. Øksendal: A Donsker delta functional approach to optimal insider control and application to finance. *Comm. Math. Stat. (CIMS)* 3 (2015), 365-421; DOI 10.1007/s40304-015-0065-y.
- [DØ2] O. Draouil and B. Øksendal: Optimal insider control and semimartingale decompositions under enlargement of filtration. arXiv: 1512.01759v1 (6 Dec.2015). To appear in *Stochastic Analysis and Applications*.
- [DiØ1] G. Di Nunno and B. Øksendal: The Donsker delta function, a representation formula for functionals of a Lévy process and application to hedging in incomplete markets. *Séminaires et Congrès, Société Mathématique de France, Vol. 16* (2007), 71-82.
- [DiØ2] G. Di Nunno and B. Øksendal: A representation theorem and a sensitivity result for functionals of jump diffusions. In A.B. Cruzeiro, H. Ouerdiane and N. Obata (editors): *Mathematical Analysis and Random Phenomena*. World Scientific 2007, pp. 177 - 190.
- [DØP] G. Di Nunno, B. Øksendal and F. Proske: *Malliavin Calculus for Lévy Processes with Applications to Finance*. Universitext, Springer 2009.
- [KMT] E. Kemajou, S.-E. Mohammed and A. Tambue: A stochastic delay model for pricing debt and loan guarantees: Theoretical results. arxiv: 1210.0570v2, 30 October 2012.
- [MØP] S. Mataramvura, B. Øksendal and F. Proske: The Donsker delta function of a Lévy process with application to chaos expansion of local time. *Ann. Inst H. Poincaré Prob. Statist.* 40 (2004), 553-567.
- [ØR] B. Øksendal and E. Røse: A white noise approach to insider trading. <http://arxiv.org.abs/1508.06376> (August 2015). To appear in T. Hida and L. Streit (editors): *Applications of hite Noise Analysis*. World Scientific, Singapore.
- [ØS1] B. Øksendal and A. Sulem: *Applied Stochastic Control of Jump Diffusions*. Second Edition. Springer 2007

- [ØS2] B. Øksendal and A. Sulem: Risk minimization in financial markets modeled by Itô-Lévy processes. *Afrika Matematika* (2014), DOI: 10.1007/s13370-014-02489-9.
- [ØSZ] B. Øksendal, A. Sulem and T. Zhang: Optimal control of stochastic delay equations and time-advanced backward stochastic differential equations. *Adv. Appl. Probab.* 42 (2011), 572-596.